S.T. YAU COLLEGE MATH COMPETITION 2014 ORAL EXAM

Algebra

Problem 1. Solve the equation $x^2 = x$ in End k^n where k is a field.
Problem 2. Let $n \geq 1$ be an integer. Construct a Galois extension over $\mathbb{Q}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Problem 3. Let $p > 3$ be a prime. Consider the equation

$$x^3 + y^3 = 1 \ (*)$$

in $\mathbb{Z}/p\mathbb{Z}$.
(1) When $p \equiv 2$ (3), find the number of solutions.
(2) When $p \equiv 1$ (3), prove that there exists a pair (a, b) of integers such that
(a) $4p = a^2 + 27b^2$
(b) $a \equiv 1 \mod 3$. (Note: a is unique.)
(3) (Continuation of (2)) When $p \equiv 1$ (3). Prove that (*) has $p-2+a$ solution in $\mathbb{Z}/p\mathbb{Z}$.